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D-Set and Groups

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Kôpka and Chovanec have defined the concept of a D-poset, a partially ordered set with a partial operation $\ominus$ with properties analogous to subtraction on the real line. In this paper we study similar structures, but we do not assume a partial order relation or the existence of distinguished elements 0, 1. We call each such structure a D-set and show that if a certain condition is satisfied, a D-set becomes the union of Abelian groups.

INTRODUCTION

Definition 1.1. Let L be nonempty set and $\ominus$ be a partial binary operation on L. Then the set L will be called a *difference set* (DS) if the following conditions are fulfilled:

(d1) For any $a \in L$, $a \ominus a \in L$ and we will denote $a \ominus a = 0_a$.
(d2) If $a, b, a \ominus b \in L$, then $a \ominus (a \ominus b) \in L$ and moreover $a \ominus (a \ominus b) = b$.
(d3) If $a, b, c, a \ominus b, b \ominus c \in L$, then $a \ominus c \in L$ and moreover $(a \ominus c) \ominus (a \ominus b) = b \ominus c$.

In the following lemma we deduce the basic properties of a DS. Similar properties were proved for difference posets in Kôpka and Chovanec (1994).

Lemma 1.1. Let L be a DS; then:

(1) For any $a \in L$, $a \ominus 0_a \in L$ and $a \ominus 0_a = a$.
(2) If $c \ominus a \in L$, then $0_a = 0_c = 0_{c\ominus a}$.
(3) If $c \ominus a = d$, then $c \ominus d = a$.
(4) If $c \ominus b$, $(c \ominus b) \ominus a \in L$, then $c \ominus a$, $(c \ominus a) \ominus b \in L$ and $(c \ominus b) \ominus a = (c \ominus a) \ominus b$.

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Proof. (1) If $a \in L$, then $a \ominus a \in L$ and $a = a \ominus (a \ominus a) = a \ominus 0_a \in L$. So $a \ominus 0_a = a$.

(2) If $c \ominus a \in L$, then $(c \ominus a) \ominus (c \ominus a) \in L$ and $0_{c\ominus a} = (c \ominus a) \ominus (c \ominus a) = a \ominus a = 0_a$. On the other hand, $c \ominus a$, $c \ominus c \in L$. Then $(c \ominus a) \ominus 0_c = (c \ominus a) \ominus (c \ominus c) = c \ominus a$ and $0_{c\ominus a} = (c \ominus a) \ominus (c \ominus a) = (c \ominus a) \ominus [(c \ominus a) \ominus 0_c] = 0_c$.

(3) If $c \ominus a \in L$ and $c \ominus a = d$, then $a = c \ominus (c \ominus a) = c \ominus d$. So $a = c \ominus d$.

(4) If $c \ominus b$, $(c \ominus b) \ominus a \in L$, then $b = c \ominus (c \ominus b)$ and $(c \ominus b) \ominus a$, $c \ominus (c \ominus b) \in L$. So we have $c \ominus a \in L$ and $(c \ominus a) \ominus b = (c \ominus a) \ominus [c \ominus (c \ominus b)] = (c \ominus b) \ominus a$. ■

Definition 1.2. Let L be a DS. The set L will be called a *group difference set* (GDS) if the following condition is satisfied:

(d4) $a \ominus b \in L$ iff $b \ominus a \in L$.

Lemma 1.2. Let L be a GDS; then:

(1) For any $a \in L$, $0_a \ominus a \in L$.
(2) For $a, b \in L$, $a \ominus b \in L$ iff $0_a = 0_b$.
(3) For $a \ominus b \in L$, $a \ominus b = 0_a \ominus (b \ominus a)$.

Proof. (1) Let $a \in L$. From the Lemma 1.1 we have $a \ominus 0_a \in L$ and from (d4) $0_a \ominus a \in L$.

(2) From Lemma 1.1 we get $a \ominus b \in L$ implies $0_a = 0_b = 0_{a\ominus b}$. On the other hand let $0_a = 0_b$. From (d4) $0_a \ominus a$, $0_b \ominus b \in L$. From axiom (d3) we get $a \ominus b \in L$.

(3) Let $a \ominus b \in L$. Then $0_a = 0_b$ and $0_a = a \ominus a = b \ominus b$. Now $0_a \ominus (b \ominus a) = (b \ominus b) \ominus (b \ominus a) = a \ominus b$. ■

Definition 1.3. Let L be a DS. If $0_b \ominus b \in L$, we define $a \oplus b := a \ominus (0_b \ominus b)$ iff $a \ominus b \in L$.

Proposition 1.3. Let L be a GDS. Then the following statements are true:

(1) The set $G(a) = \{b \in L: 0_a = 0_b\}$ is an Abelian group, with the given operation.
(2) If for any $a, b \in L$, $0_a = 0_b$, then L is an Abelian group with the given operation.

Proof. (1) From Lemma 1.2 we know that $0_a = 0_b$ iff $a \ominus b$, $b \ominus a \in L$. If $a \in L$, then $a \oplus 0_a = a \ominus (0_a \ominus 0_a) = a$.

Now we show the commutative law. Let $b \oplus a \in L$. Then $0_a = 0_b$ and we can calculate $b \oplus a = b \ominus (0_a \ominus a) = (0_a \ominus (0_a \ominus b)) \ominus (0_a \ominus a) = a \ominus (0_a \ominus b) = a \oplus b$.

The associative law: Let $(a \oplus b) \oplus c \in L$; then $a, b, c \in G(a)$. This implies $(a \oplus b) \oplus c = (a \oplus b) \ominus (0_a \ominus c) = [a \ominus (0_a \ominus b)] \ominus (0_a \ominus c) = [a \ominus (0_a \ominus c)] \ominus (0_a \ominus b) = (a \oplus c) \oplus b$. Moreover, $(a \oplus b) \oplus c) = (a \oplus c) \oplus b = (c \oplus a) \oplus b = (c \oplus b) \oplus a = a \oplus (c \oplus b)$.

Let $b, d \in G(a)$ and let there be two elements $x_1, x_2 \in G(a)$ such that $d \oplus x_1 = d \oplus x_2 = b$. If $b = d \oplus x_1$, then $b = d \ominus (0_a \ominus x_1)$. From this we get $0_a \ominus x_1 = d \ominus b$ and moreover $x_1 = b \ominus d$. On the other hand, $x_2 = b \ominus d$. This implies $x_1 = x_2$. Thus $G(a)$ is Abelian group.

(2) Suppose that for all $a, b \in L$, $0_a = 0_b$. Then from Lemma 1.2 we have $a \ominus b$, $b \ominus a \in L$ and for every $a, b \in L$, $G(a) = G(b) = L$. ∎

Proposition 1.4. L is a GDS if it can be written as disjoint union of Abelian groups. Conversely, every such disjoint union is a GDS.

Proof. Let L be a GDS. If $G(a) = \{b \in L\colon 0_a = 0_b\}$, then $G(a)$ is an Abelian group and it is clear that $0_a = 0_b$ iff $G(a) = G(b)$ and $0_a \neq 0_b$ iff $G(a) \cap G(b) = \emptyset$. For any $c \in L$, $c \in G(c)$ and then

$$L \subseteq \bigcup_{c \in L} G(c)$$

On the other hand, for any $a \in L$, $G(a) \subseteq L$. From this it follows that

$$L = \bigcup_{a \in L} G(a)$$

Conversely, let $\{T_\alpha\}_{\alpha \in \Gamma}$ be a family of disjoint Abelian groups. Let $\mathcal{T} = \bigcup_{\alpha \in \Gamma} T_\alpha$. We can define a partial binary operation $\oplus$ as follows: $a \oplus b \in \mathcal{T}$ iff there exists $\alpha \in \Gamma$ deal with $a, b \in T_\alpha$ and moreover $a \oplus b = a \oplus_\alpha b$, where $\oplus_\alpha$ is the group operation on T_α. In the following we will denote by the symbol a^- such an element from $\mathcal{T}$ that if $a \in T_\alpha$, then $a^- \in T_\alpha$ deal with $a \oplus a^- = 0_\alpha$. Let a partial binary operation $\ominus$ on $\mathcal{T}$ be defined as follows: $a \ominus b \in \mathcal{T}$ iff there exist $\alpha \in \Gamma$ with $a, b \in T_\alpha$ and $a \ominus b = a \oplus b^-$. In the following we show that $\mathcal{T}$ is a GDS.

If $a \in \mathcal{T}$, then there is $\alpha \in \Gamma$ such that $a \in T_\alpha$ and $0_\alpha = a \oplus a^- = a \ominus a = 0_a \in \mathcal{T}$.

If $a, b \in \mathcal{T}$ and $a \ominus b \in \mathcal{T}$, then there is $\alpha \in \Gamma$ such that $a, b \in T\alpha$. Now $a \ominus b \in \mathcal{T}$ and $a \ominus b = a \oplus_\alpha b^-$. But T_α is an Abelian group. Then $a^- \in T_\alpha$ and $b \oplus_\alpha a^- \in T_\alpha$. And $b \oplus a^- = b \ominus a \in \mathcal{T}$ and $0_a = 0_b = 0_\alpha$.

If $a \ominus b \notin \mathcal{T}$, then $a \oplus b^- \notin T$. This means that there are $\alpha, \beta \in \Gamma$ with $a \in T_\alpha$, $b \in T_\beta$, and $T_\alpha \cap T_\beta = \emptyset$. From this it follows that $b \ominus a \notin \mathcal{T}$. We conclude that $a \ominus b \in \mathcal{T}$ iff $a \ominus a \in \mathcal{T}$.

Let $a \ominus b \in \mathcal{T}$. Then there exists $\alpha \in \Gamma$ with $a \ominus b = a \oplus_\alpha b^- \in T_\alpha$. Hence T_α is an Abelian group; then $a \oplus_\alpha (a \oplus_\alpha b^-)^- \in T_\alpha$. Thus

$$\begin{aligned}(a \oplus_\alpha b^-) &\oplus_\alpha (a^- \oplus_\alpha b)\\ &= ((a \oplus_\alpha b^-) \oplus_\alpha a^-) \oplus_\alpha b\\ &= ((a \oplus_\alpha s^-) \oplus_\alpha b^-) \oplus_\alpha b\\ &= (0_a \oplus_\alpha b^-) \oplus \alpha b\\ &= b^- \oplus_\alpha b\\ &= 0_a\end{aligned}$$

and

$$(a \oplus_\alpha b^-) \oplus_\alpha (a \oplus_\alpha b^-)^- = 0_a$$

From the fact that T_α is an Abelian group we get

$$(a \oplus_\alpha b^-)^- = a^- \oplus_\alpha b$$

This implies

$$a \ominus (a \ominus b) = a \oplus_\alpha (a \oplus_\alpha b^-)^- = a \oplus_\alpha (a^- \oplus_\alpha b) = (a \oplus_\alpha a^-) \oplus_\alpha b = b$$

which implies

$$a \ominus (a \ominus b) = b$$

Let $a \ominus b$, $b \ominus c \in \mathcal{T}$. Then there exist α, $\beta \in \Gamma$ such that $a \ominus b \in T_\alpha$ and $b \ominus c \in T_\beta$. From the definition of the partial binary operation $\ominus$ we get $a, b \in T_\alpha$, $b, c \in T_\beta$, and then $T_\alpha \cap T_\beta \neq \emptyset$. This implies that $T_\alpha = T_\beta$, and $k \oplus_\alpha r = k \oplus_\beta r$ for every $k, r \in T_\alpha$. From this it follows that $a \ominus c = a \oplus_\alpha c^- \in T_\alpha$, $(a \ominus c) \ominus (a \ominus b) \in T_\alpha$, and moreover

$$\begin{aligned}(a \ominus c) &\ominus (a \ominus b)\\ &= (a \oplus_\alpha c^-) \oplus_\alpha (a \oplus_\alpha b^-)^-\\ &= (a \oplus_\alpha c^-) \oplus_\alpha (a^- \oplus_\alpha b)\\ &= (a \oplus_\alpha (a^- \oplus_\alpha b)) \oplus_\alpha c^-\\ &= ((a \oplus_\alpha a^-) \oplus_\alpha c^-\\ &= b \oplus_\alpha c^-\\ &= b \ominus c\end{aligned}$$

Thus $\mathcal{T}$ is a GDS. ∎

Corollary 1.4.1. Let L be a GDS. The set L is an Abelian group iff for every $a, b \in L$, $0_a = 0_b$.

Proof. For every $a, b \in L$, $0_a = 0_b = 0$, implies $L = G(a) = G(b)$. Hence $G(a)$ is an Abelian group.

If L is an Abelian group, then for every $a, b \in L$, $a \ominus b = a \oplus b^- \in L$. This implies that $0_a = 0_b = 0$. ■

Definition 1.4. Let L be a nonempty set with a partial binary operation $\ominus$. We call a subset E_0 of L an *ordering set* if the following conditions are fulfilled:

(1) If $a, 0_a \ominus a \in L$ and $0_a \ominus a \neq a$, then $a \in E_0$ iff $0_a \ominus a \notin E_0$.
(2) If $a \ominus b, b \in E_0$, then $a \in E_0$.

Lemma 1.5. Let L be a GDS with the partial ordering set E_0. Let $H = \{a \ominus b: a \ominus b = b \ominus a\}$. Then $E_0 \cap H = \emptyset$.

Proof. Let there be element $a \in L$ such that $a \neq 0_a$ and $a \in E_0 \cap H$. If $a \in H$, then $a = 0_a \ominus a$. But if $a \in E_0$, from condition (1), $a \in E_0$ iff $0_a \ominus a \notin E_0$. So $0_a \ominus a \notin E_0$. This contradicts $a = 0_a \ominus a$. Hence $E_0 \cap H = \emptyset$. ■

Proposition 1.6. Let L be a GDS. If there exist a partial ordering set E_0 and we write $a \preceq b$ iff (1) $a \prec b$ iff $b \ominus a \in E_0$ or (2) $a \sim b$ iff $a \ominus b = b \ominus a$, then $\preceq$ is a partial ordering on L.

Proof. Let H be as Lemma 1.5. Then we know that $H \cap E_0 = \emptyset$. Hence $a \ominus b \in L$ iff $\{a \ominus b, b \ominus a\} \cap E_0 \neq \emptyset$ or $a \ominus b \in H$. It is clear that $0_a \in H$ and hence $a \sim a$.

Let $a \preceq b$ and $b \preceq a$. If $a \ominus b \in H$, then $b \ominus a \in H$, so that $a \sim b$.

Let $a \preceq b$, $b \preceq c$. We want to show that $a \preceq c$. From the definition for the relation $\preceq$ there exist the following possibilities:

(1) $a \prec b$ and $b \prec c$.
(2) $a \sim b$ and $b \sim c$.
(3) $a \sim b$ and $b \prec c$.
(4) $a \prec c$ and $c \sim b$.

Because $0_a = 0_b$, $0_b = 0_c$, then $0_a = 0_c$ and $c \ominus a$, $a \ominus c \in L$.

(1) If $a \prec b$ and $b \prec c$, then $b \ominus a$, $c \ominus b \in E_0$. Hence L is a GDS; then

$$(c \ominus a) \ominus (c \ominus b) = b \ominus a$$

From the definition of the set E_0 it follows that $c \ominus a \in E_0$, which implies that $a \prec c$.

(2) If $a \sim b$ and $b \sim c$, then $a \ominus b = b \ominus a$ and $c \ominus b = b \ominus c$. Hence

$$(c \ominus b) \ominus (c \ominus a) = a \ominus b$$

and from this follows $c \ominus a = (c \ominus b) \ominus (a \ominus c) = (b \ominus c) \ominus (b \ominus a) = a \ominus c$. This implies $a \ominus c \in H$. We conclude that $a \sim c$.

(3) If $a \sim b$ and $b < c$, then $a \ominus b \in H$ and $c \ominus b \in E_0$. Let $c \sim a$. Because $(c \ominus a) \ominus (c \ominus b) = b \ominus a$, we have $c \ominus b = (c \ominus a) \ominus (b \ominus a) = (a \ominus c) \ominus (a \ominus b) = b \ominus c$. But this means that $b \ominus c \in H$. This contradicts the assumption and hence $a \ominus c \neq c \ominus a$. Let $a \ominus c \in E_0$. Then from the basic property we get

$$(a \ominus b) \ominus (a \ominus c) = c \ominus b$$

From the definition of the set E_0, $c \ominus b$, $a \ominus c \in E_0$ implies $a \ominus b \in E_0$. This contradicts the assumption. Hence $c \ominus a \in E_0$ and consequently $a < c$.

(4) Let $a < b$ and $b \sim c$; then $a \ominus b = b \ominus a$. It is clear that $(c \ominus a) \ominus (c \ominus b) = b \ominus a$. If $c \ominus a \in H$, then $c \ominus a = a \ominus c$ and we can calculate $b \ominus a = (c \ominus a) \ominus (c \ominus b) = (a \ominus c) \ominus (b \ominus c) = (0_a \ominus (c \ominus a)) \ominus (0_a \ominus (c \ominus b)) = (c \ominus b) \ominus (c \ominus a) = a \ominus b$. This implies that $b \ominus a \in H \cap E_0 = \emptyset$. Hence $c \ominus a \neq a \ominus c$. Let $a \ominus c \in E_0$; then

$$(b \ominus c) \ominus (b \ominus a) = a \ominus c$$

From the definition of the set E_0 we get $b \ominus c \in E_0$. This contradicts the assumption. Hence $c \ominus a \in E_0$. This implies $a < c$. ■

Now it is clear that if L is a GDS such that $0_a = 0_b$ for all $a, b \in L$, then L is an Abelian group. From this it follows that if we have L as a D-poset (D set with the partial ordering) and we assume that for any $a \ominus b \in L$, $b \ominus a \in L$, we get the ordering group. If L is a GDS and if there is an element $a \in L$ such that $G(a) \cap E_0 \neq \emptyset$, then we can define partial ordering on the difference set L.

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